

# Math 121 Practice Midterm Solutions

October 14, 2014

## 1 Exam Solutions

**Problem 1.1.** (True/False) Most of these should be clear.

1. True.
2. False; Consider the case where  $A_1 = A_2 = \mathbb{R}^2$ . Can write different bases for  $\mathbb{R}^2$  such that the intersection does not form a basis for  $A_1 \cap A_2 = \mathbb{R}^2$ .
3. False; Take the top degree coefficient of  $g$  to be the negation of the top degree coefficient of  $f$ .
4. True.
5. True.
6. True.
7. False.

**Problem 1.2.** Let  $\alpha : U \rightarrow V$  and  $\beta : V \rightarrow W$  be surjective. We know that for all  $w \in W$ , there is a  $v \in V$  such that  $\beta(v) = w$ . This implies the existence of  $u \in U$  such that  $\alpha(u) = v$ .

Then, by what it means to be a composition,  $\beta \circ \alpha(u) = \beta(\alpha(u)) = w$ . Hence, the composition is surjective.

**Problem 1.3.** Suppose that  $(v_1, \dots, v_k)$  is linearly dependent. Then by the Linear Dependence Lemma, there exists a  $j \in \{2, \dots, k\}$  such that  $v_j \in \text{span}(v_1, \dots, v_{j-1})$  and  $(v_1, \dots, v_{j-1})$  linearly independent. Thus, there are scalars  $a_1, \dots, a_{j-1} \in \mathbb{F}$  such that

$$v_j = a_1 v_1 + \dots + a_{j-1} v_{j-1}. \quad (1.1)$$

Apply  $T$  to both sides, and note that  $v_i$  is an eigenvector with eigenvalue  $\lambda_i$  to see that

$$\lambda_j v_j = a_1 \lambda_1 v_1 + \dots + a_{j-1} \lambda_{j-1} v_{j-1}. \quad (1.2)$$

Subtract  $\lambda_j(v_j = a_1 v_1 + \dots + a_{j-1} v_{j-1})$  from the above to get

$$0 = (\lambda_j - \lambda_1) a_1 v_1 + \dots + (\lambda_j - \lambda_{j-1}) a_{j-1} v_{j-1}. \quad (1.3)$$

Since  $(v_1, \dots, v_{j-1})$  is linearly independent, we must have  $(\lambda_j - \lambda_i) a_i = 0$  for all  $i \in \{1, \dots, j-1\}$ . We assumed all eigenvalues distinct, so  $\lambda_j - \lambda_i \neq 0$ , which implies that  $a_i = 0$  for all  $i \in \{1, \dots, j-1\}$ . But then, by (1.1),  $v_j = 0$  which contradicts the assumption that all eigenvectors are nonzero. Hence,  $(v_1, \dots, v_k)$  is linearly independent.

**Problem 1.4.** 1. A basis for the kernel (null) is  $T$  is simply  $\{v_1 - v_2\}$ .

2. A basis for the image (range) of  $T$  is simply  $\{w_1, w_2\}$ .
3. Rank-Nullity says that, for  $V$  finite dimensional,  $\dim V = \dim(\text{null } T) + \dim(\text{range } T)$ .
4. Indeed, our answers are consistent. We have that the null space is 1-dimensional and the range is 2-dimensional, which sum to the dimension of  $V$ .

**Problem 1.5.** We have that  $F : P_n(\mathbb{R}) \rightarrow P_{n-1}(\mathbb{R})$  is explicitly given by  $u \mapsto u'$ . Taking the standard bases  $\alpha := \{1, \dots, t^n\}$  for  $P_n(\mathbb{R})$  and  $\beta := \{1, \dots, t^{n-1}\}$  for  $P_{n-1}(\mathbb{R})$  we wish to write  $M(F, \alpha, \beta)$ . Simply look at what the transformation does to our basis vectors.

Consider  $n = 2$ . We have:  $F(1) = 0, F(t) = 1, F(t^2) = 2$ . Thus, we write

$$M(F, \alpha, \beta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

We proceed in this way for larger  $n$ , with the super-diagonal filling to  $n - 1$ .

**Problem 1.6.** This is early just citing definitions and results, which I assume you all can do!